Multiresolution analysis & wavelets (quick tutorial)

Application: image modeling

André Jalobeanu
Multiresolution analysis

\[ \Lambda = \text{support (R, R}^2, \ldots) \]

Set of closed nested subspaces \( \left( V_j \right)_{j \in \mathbb{Z}} \) of \( L^2(\Lambda) \)

\[ \ldots V_3 \subset V_2 \subset V_1 \subset V_0 \subset \ldots \]

\( j = \text{scale, resolution} = 2^{-j} \) (dyadic wavelets)

Approximation \( a_j \) at scale \( j \) : projection of \( f \) on \( V_j \)

\( \left( \phi_{jl} \right)_{l \in \Omega} \) Basis of \( V_j \) at scale \( j \); \( l = \text{spatial index} \)
Multiresolution decomposition

Set of approximations and details

\[ a_j \quad d_j^k \quad k = \text{subband index (orientation, etc.)} \]

\[ (V_j)_{j \in \mathbb{Z}} \quad (W_j^k)_{j \in \mathbb{Z}} \]

\[ V_{j-1} = V_j \oplus (W_1^j \oplus \ldots \oplus W_n^j) \]

\[ (\psi_{kjl})_{l \in \Omega} \quad \text{Basis of } W_j^k \text{ at scale } j ; \quad l = \text{spatial index} \]

= wavelets
Space / Frequency representation (wavelet basis functions)

Compromise between spatial and frequential localization

uncertainty principle

…different wavelet shapes
1D wavelet basis

Wavelet $\psi$:
- \( L^2(\mathbb{R}) \)
- \( \| \psi \|^2 = 1 \)
- zero mean

Dilations / shifts:
\[
\psi_{j,n}(t) = \frac{1}{\sqrt{2^j}} \psi(2^{-j}t - n)
\]

Scale function $\phi$

Multiresolution analysis [Mallat]
\[
\phi_{j,n}(t) = \frac{1}{\sqrt{2^j}} \phi(2^{-j}t - n)
\]

Basis of $L^2(\mathbb{R})$

Basis of $V_j$: approximation at res. $2^{-j}$
2D tensor product wavelet basis

approximations

\[ \phi_{j,n,m}(x,y) = \phi_{j,n}(x)\phi_{j,m}(y) \]

\[
\begin{align*}
\psi_{j,n,m}^1(x,y) &= \phi_{j,n}(x)\psi_{j,m}(y) \\
\psi_{j,n,m}^2(x,y) &= \psi_{j,n}(x)\phi_{j,m}(y) \\
\psi_{j,n,m}^3(x,y) &= \psi_{j,n}(x)\psi_{j,m}(y)
\end{align*}
\]
2D Wavelet transform using filter banks

In practice: **discrete wavelet transform** [Mallat, Vetterli]

\[ \phi \text{ et } \psi \text{ completely defined by the discrete filters } h \text{ and } g \]

\[ (a,d^1,d^2,d^3) \text{ at scale } 2^{-j} \rightarrow (a,d^1,d^2,d^3) \text{ at scale } 2^{-j-1} \]
Wavelet transform tree

- $j=0$
- $j=1$
- $j=2$
- $j=3$
Wavelet packet transform tree

→ decompose the detail subbands
[Mallat]
Wavelet packet basis

approximations

\[ \phi_{j,n,m}(x,y) = \phi_{j,n}(x)\phi_{j,m}(y) \]

details

\[ \psi_{j,n,m}^{p,q}(x,y) = \psi_{j,n}^{p}(x)\psi_{j,m}^{q}(y) \]

Wavelet packets
Complex wavelet packets

Properties:

- Shift invariance
- Directional selectivity
- Perfect reconstruction
- Fast algorithm $O(N)$

- **quad-tree** (4 parallel wavelet trees)  [Kingsbury 98]
- filters **shifted** by $\frac{1}{2}$ and $\frac{1}{4}$ pixel between trees
- combination of trees $\rightarrow$ **complex** coefficients
- **biorthogonal** wavelets
- **filter bank** implementation
Quad-tree: 1st level

Non-decimated transform

Parallel trees ABCD

Perfect reconstruction: mean \((A+B+C+D)/4\)
Quad-tree: level $j$

different length filters: $h^o, g^o, h^e, g^e \rightarrow$ shift < pixel
Frequency plane partition
Directional selectivity
impulse responses – real part

Complex wavelets

Complex wavelet packets
Why use wavelets ?
Self-similarity of natural images: P1 (1)

- **IMAGE**
- **Spectrum**
  - Energy $w$
  - Radial frequency $r$

**Power spectrum decay?**

$log w$

$log r$
Self-similarity of natural images (2)

scale invariance

or self-similarity

\[ w = w_0 \, r^{-q} \]

Power spectrum decay

\[ \log w \quad \log r \]
Non-stationarity of natural images: P2

2. Modélisation des images

- textures
- smooth areas
- small features
Image modeling

- Fractional brownian motion \((w_0, q)\)
  - Fractal model
- Non-stationary multiplier function
- Wavelet transform \(\rightarrow \sim\) independent coefficients \((\sim K-L)\)
- Subband histogram
- Heavy-tailed distribution
Inter-scale dependence

Wavelet transform

Inter-scale persistence of the details
Basis choice (1)

Optimal representation of features by different wavelet shapes

Sparse representation: keep a small number of coefficients

Asymptote $E \sim N^{-1/2}$

Haar
Symmlet-8
Basis choice (2) : invariance properties

Shift invariance ?

Rotation invariance ?
Wavelet zoo

- Orthogonal wavelets
- Biorthogonal wavelets
- Non-decimated (redundant) decompositions
- Pyramidal representations (Burt-Adelson, etc.)
- Wavelets-vaguelettes (deconvolution)
- Non-linear multiscale transforms (lifting, non-linear prediction)
- Curvelet transform (better represents curves)
- Complex wavelets
- Non-separable wavelets
- Wavelets on manifolds
- …